CHARACTERISTICS OF THE NUMERICAL SOLUTION OF THE HEAT-CONDUCTION EQUATION AT EARLY TIMES

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A criterion is presented for choosing time steps Δt and space steps Δx in an implicit scheme for obtaining a numerical solution of the unsteady heat-conduction equation at early times.

The problem under consideration is formulated as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} + b\dfrac{\partial u}{\partial x} + cu, & (1) \\ u(0,x) = 1, & \\ u(t,0) = 0,\ \dfrac{\partial u}{\partial x}(t,+\infty) = 0. & (1a) \end{cases}$$

The general finite difference approximation of Eq. (1) has the form

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = a\frac{\theta(\delta^2 u)_j^{n+1} + (1-\theta)(\delta^2 u)_j^n}{\Delta x^2} + b\frac{\theta(u_{j+1}^{n+1} - u_{j-1}^{n+1}) + (1-\theta)(u_{j+1}^n - u_{j-1}^n)}{2\Delta x} + cu_j^n, \tag{2}$$

where $1/2 \leqslant \theta \leqslant 1$; δ^2 is the centered second difference operator, and the subscripts and superscripts denote distance and time, respectively. The von Neumann method for investigating stability [1] gives the amplification factor

$$\lambda = \frac{1 - 4ar(1-\theta)\sin^2\frac{\varphi}{2} + c\Delta t + ib(1-\theta)\sin\varphi\sqrt{\Delta tr}}{1 + 4ar\theta\sin^2\frac{\varphi}{2} - ib\theta\sin\varphi\sqrt{\Delta tr}}, \tag{3}$$

where $\varphi = k\Delta x$ and $r = \Delta t/\Delta x^2$.

By requiring that $|\lambda| \leqslant 1 + O(\Delta t)$ we find that the finite difference scheme is unconditionally stable for $1/2 \leqslant \theta \leqslant 1$; i.e., all the components of the initial perturbation are damped out with time. A detailed proof of this statement is given in [2].

By introducing the notation $\psi = 4r\sin^2(\varphi/2)$ we can write the real part of the amplification factor in the form

$$\mathrm{Re}(\lambda) = \frac{a^2\theta(\theta-1)\Psi^2 - [a + b^2\theta(1-\theta)\cdot\Delta t\cos^2\frac{\varphi}{2} - ca\theta\Delta t - 2a\theta]\Psi + c\Delta t + 1}{(1 + a\theta\Psi)^2 + b^2\theta^2\Delta t\cos^2\frac{\varphi}{2}\Psi}. \tag{4}$$

If $\Psi \to +\infty$, $\mathrm{Re}(\lambda) \to -(1-\theta)/\theta$ (Fig. 1). Hence, it follows that only for $\theta = 1$ is $\mathrm{Re}(\lambda) > 0$ for all values of Ψ. If $\theta < 1$, a value Ψ_0 can always be found such that, for $\Psi > \Psi_0$, $-1 < \mathrm{Re}(\lambda) \leqslant 0$. Thus, the damping of the initial perturbation is oscillatory in character, and this decreases the accuracy of the calculation for early times. Equating the numerator of (4) to zero and performing some simple calculations, we obtain the condition for r which

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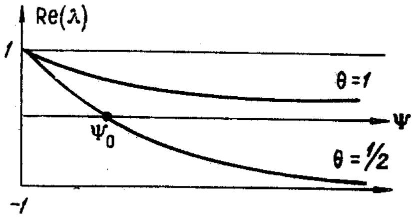

Fig. 1. Real part of amplification factor λ as a function of Ψ.

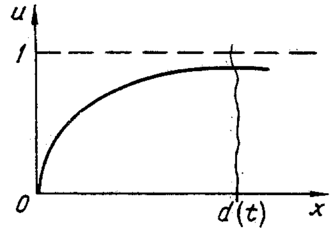

Fig. 2. Boundary-layer region for the heat-conduction equation.

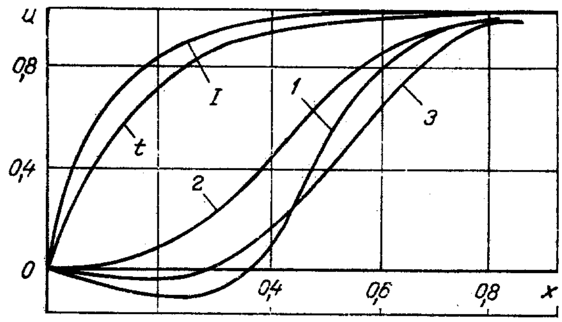

Fig. 3. Dimensionless temperature u as a function of distance x/d(t); I, comparison with exact solution; 1, 2, and 3, temperature distribution over first, second, and third time slices, respectively, for parameters not satisfying (5). $\Delta t = 10^{-5}$, $d = 10^{-3}$, $\theta = 0.7$.

ensures damping of the initial perturbation without oscillations for all frequencies of the expansion of the solution in a Fourier series

$$4r \leqslant \frac{1}{a(1-\theta)} + \frac{c\Delta t}{a(1-\theta)} - \frac{b^2\theta\Delta t}{2a^2}. \tag{5}$$

Thus, it is clear that although the stability condition permits the choice of any relation between Δt and Δx, condition (5) imposes restrictions on r. The larger a and b^2/a^2 the more stringent the restrictions on Δt for a chosen 0x. If $c < 0$ this also imposes further restrictions on Δt. We note that for $c = b = 0$, $\theta = 1/2$, $a = 1$, we obtain the relation $r \leqslant 1/2$ recommended in [3].

A test of this criterion was performed on the solution of the heat-conduction equation using the scheme proposed by Paskonov [4]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \\ u(0, x) = 1, \\ u(t, 0) = 0, \quad \dfrac{\partial u}{\partial x}(t, +\infty) = 1. \end{cases} \tag{6}$$

Condition (5) in this case is written in the form

$$4r \leqslant \frac{d^2(t)}{1-\theta} - \Theta d_t'(t)\,\Delta t d^2,$$

where d(t) is the boundary-layer region for the heat-conduction equation (Fig. 2).

Figure 3 shows the solution of problem (6) at early times for various Δt. It can be seen from the curves that condition (7) is essential to ensure the necessary accuracy at early times.

It should be noted that it is particularly important to satisfy inequality (5) in solving nonlinear problems. Thus, the solution of a nonlinear heat-conduction problem showed that the iterations do not converge if inequality (5) is not satisfied, where a, b, and c are taken as the extreme values of the corresponding coefficients of the equation.

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THE STRESS-DIFFUSION EFFECT IN HETEROGENEOUS LIQUID STREAMS

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An analysis is made of the effect of the shear stresses in the average stream on the coefficient of molar diffusion of a passive impurity in a turbulent liquid stream.

In a study of the slow motion of a specularly reflecting sphere in a heterogeneous stream of rarefied gas a dependence of the coefficient of dynamic friction on the tensor of shear stress in the stream (the stress-phoretic effect) was obtained in [1]. In this case the transfer processes are determined by the following tensor expression for the coefficient of Brownian diffusion:

$$D_{ij} = D^0(\delta_{ij} - \alpha\sigma_{ij}/p), \tag{1}$$

where D^0 is the Einstein coefficient of Brownian diffusion; σ_{ij} are the coefficients of the reduced stress tensor; p is the hydrostatic pressure; δ_{ij} is the Kronecker symbol; α is a numerical coefficient. Analogous results follow within the framework of the kinetic theory in an analysis of the motion of heavy impurity particles in a nonequilibrium gas [2].

In the present report it is shown that a similar dependence of the diffusion coefficient on the stress tensor in a stream also occurs in the case of the turbulent diffusion of a passive impurity in a shear stream (the stress-diffusion effect).

Let $v(t, x; \omega)$ be the random vector field of the velocity of a turbulent stream of incompressible liquid; $\vartheta(t, x; \omega)$ be the scalar field of the concentration of a passive impurity in it, which satisfies the transfer equation [3]

$$\frac{\partial}{\partial t}\vartheta(t, x; \omega) = -\frac{\partial}{\partial x_i} v_i(t, x; \omega)\,\vartheta(t, x; \omega). \tag{2}$$

Let us examine the derivation of the equation of convective diffusion for $\bar{\vartheta}$ on the assumption of the statistical independence of the initial distribution $\vartheta(0, x; \omega) = \vartheta_0(x; \omega)$ from the velocity fluctuations in the in the turbulent stream and in the approximation of a Gaussian velocity field. We take $v_i = \bar{v}_i + v'$ and $\bar{v}_i' = 0$ and introduce the designation

$$\Omega(t; \omega) = -U^{-1}(t)\frac{\partial}{\partial x_i} v_i'(t, x; \omega)\,U(t),\; U(t) = \exp\left[-t\frac{\partial}{\partial x_i}\bar{v}_i(x)\right], \tag{3}$$

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